REFLECTION, SCATTERING, AND ABSORPTION

OF ACOUSTIC WAVES BY ROUGH SURFACES

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Abstract

The first and second moments, i.e. the coherent field and the two-point two-time correlation function, are calculated for the acoustic fields scattered from various rough surfaces. For each surface ~~they~~ these yield the reflection, absorption and differential scattering coefficients, as well as an equivalent boundary condition for the coherent field. Renormalized coefficients are constructed to eliminate divergences at grazing incidence. The results are specialized to surfaces which are statistically homogeneous in both space and time, to surfaces which are not moving, to surfaces which are simply or multiply periodic, and to surfaces consisting of randomly placed bosses on a smooth surface. The surfaces considered are slightly rough, moving, soft or hard boundaries, and flat surfaces with random admittances or impedances. The analysis is based on the regular perturbation method or Born approximation. Comparisons with previous results are made.

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1. Introduction.

When a wave is incident upon a rough surface, it produces reflected, transmitted, and scattered waves. The energy flux in each wave can be determined by the transport theory of radiation. This theory uses geometrical acoustics to describe energy propagation, together with phenomenological coefficients to describe reflection, transmission or absorption, and differential scattering at the surface. However it does not determine these coefficients in terms of the properties of the surface and it does not describe the fluctuation and correlation properties of the waves.

Several methods of analysis, based upon the wave equation, have been devised to overcome these two shortcomings. We shall employ one of them, the regular perturbation method or Born approximation, to calculate the first and second moments of the field. These moments represent the coherent field and the two-point two-time correlation function of the field, so they yield information about field fluctuation and correlations. From them we shall obtain expressions for the various coefficients which occur in transport theory. In addition, we shall show that the two-point two-time correlation function of the field can be expressed in terms of these same coefficients.

Since the Born approximation has been used to analyze reflection by rough surfaces, the present work may be viewed as a completion of that analysis. It treats in a unified way four different cases: a nearly flat soft rough surface, a nearly flat hard rough surface, a flat surface with a small random impedance, and a flat surface with

a small random admittance. The rough surface is allowed to have a small random velocity, and the random admittance and impedance are allowed to vary in time. The first and second moments of the field are obtained in all these cases, which has not been done before. Furthermore the coefficients of reflection, etc., are modified or renormalized to eliminate divergences which occur at grazing incidence.

From the general results, more explicit ones are obtained for a number of special kinds of surface roughness. First surfaces which are statistically homogeneous or stationary in space and time are considered. Then surfaces composed of bosses randomly located on a flat surface are treated. These are the surfaces which have been studied extensively by Twersky [1] using a very different method of analysis, so we compare our results with his. In this way we are able to relate the results of the perturbation method to those of Twersky's self-consistent field method.

A survey of the theory of scattering from rough surfaces together with references, is contained in pages 54-73 of the review article by DeSanto [2].

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2. Formulation and solution in the nonrandom case.

Let $\psi(x,y,z,t)$ and $\phi(x,y,z,t)$ be the incident and scattered acoustic velocity potentials in a uniform medium of density ρ and sound speed c. We assume that ψ is a given incoming solution of the wave equation near a bounding surface, while everywhere above this surface ϕ is an outgoing solution of the wave equation:

$$(\Delta - c^{-2}\partial_t^2)\phi = 0 \ . \tag{1}$$

We shall consider four different boundary value problems for (1) and denote them a, b, c, d. The first two deal with a nearly flat, moving surface $z = \varepsilon h(x,y,t)$. Here ε is a small parameter and h is a given function.

Case a represents a soft or pressure release surface on which the pressure vanishes,

$$\rho\partial_t(\psi + \phi) = 0 \text{ on } z = \varepsilon h \ . \tag{2a'}$$

Case b represents a hard or rigid surface on which the normal component of acoustic velocity equals the normal velocity of the surface,

$$(\partial_z - \varepsilon h_x\partial_x - \varepsilon h_y\partial_y)(\psi + \phi) = \varepsilon h_t \quad \text{on } z = \varepsilon h \ . \tag{2b'}$$

The right side of (2b') is an inhomogeneous or source term for the radiation produced by the moving surface. We shall not consider this radiation, so we shall investigate the homogeneous form of (2b'), which corresponds to a vanishing normal component of acoustic velocity on the surface.

In addition to non-flat surfaces, we shall consider the flat surface $z = 0$. Case c involves the admittance boundary condition

$$(\varepsilon h \partial_t + c\partial_z)(\psi + \phi) = 0 \text{ on } z = 0 \ . \tag{2c}$$

In case d we have the impedance boundary condition

$$(\partial_t + \varepsilon hc \partial_z)(\psi + \phi) = 0 \text{ on } z = 0 \ . \tag{2d}$$

The specific admittance in (2c) and the specific impedance in (2d) are both equal to εh.

To solve these four problems we first expand $\phi + \psi$ in (2a'), and in the homogeneous form of (2b'), in a Taylor series in z about $z = 0$, and set $z = \varepsilon h$ to obtain

$$[1 + \varepsilon h\partial_z + \frac{\varepsilon^2 h^2}{2} \partial_z^2 + 0(\varepsilon^3)]\partial_t(\phi + \psi) = 0 \text{ on } z = 0 \ , \tag{2a}$$

$$\begin{aligned}[\partial_z &+ \varepsilon(h\partial_z^2 - h_x\partial_x - h_y\partial_y) + \varepsilon^2(\frac{h^2}{2} \partial_z^3 - hh_x\partial_x\partial_z - hh_y\partial_y\partial_z) \\ &+ 0(\varepsilon^3)](\phi + \psi) = 0 \text{ on } z = 0 \ .\end{aligned} \tag{2b}$$

Next we introduce the vectors $p = (x,y,t)$ and $s = (\alpha,\beta,\omega)$ and the Fourier transform pair

$$F(s) = (2\pi)^{-3} \int f(p)e^{-ip\cdot s}dp \ , \tag{3}$$

$$f(p) = \int F(s)e^{ip\cdot s}ds \ . \tag{4}$$

We shall use the corresponding capital letter to denote the transform of a function represented by a small letter. When $f(p)$ is real we have $F(-s) = \overline{F(s)}$ where the overbar denotes the complex conjugate.

Upon applying the transform (3) to (1) we find that $\Phi(s,z)$, the transform of the scattered field, satisfies

$$[\partial_z^2 + k^2(s)]\Phi(s,z) = 0 \ . \tag{5}$$

Here k^2 is defined by

$$k^2(s) = \omega^2/c^2 - \alpha^2 - \beta^2 \ . \tag{6}$$

The branch of the square root used in defining $k(s)$ is

$$\begin{aligned} k(s) &= (\omega/c)\left[1 - \frac{(\alpha^2 + \beta^2)}{\omega^2/c^2}\right]^{1/2} , \quad (\omega/c)^2 \geq \alpha^2 + \beta^2 , \\ &= -i(\alpha^2 + \beta^2 - \omega^2/c^2)^{1/2} , \quad (\omega/c)^2 \leq \alpha^2 + \beta^2 . \end{aligned} \tag{7}$$

The solution of (5) which is bounded at $z = +\infty$ when k is imaginary, or which is outgoing when k is real, is of the form

$$\Phi(s,z) = A_s(s)e^{-ik(s)z} \ . \tag{8}$$

We assume that the transform $\Psi(s,z)$ of the incident field has the form

$$\Psi(s,z) = A_i(s)e^{ik(s)z} \ . \tag{9}$$

From (8) and (9) we see that the propagating modes, those for which $(\omega/c)^2 > \alpha^2 + \beta^2$, are incoming in Ψ and outgoing in Φ. Similarly the evanescent modes in Φ decay with increasing distance z from the boundary, while those in Ψ grow with z. Therefore the representation (9) of Ψ cannot hold for z arbitrarily large if it contains evanescent modes.

The amplitude $A_i(s)$ in (9) is supposed to be known while the amplitude $A_s(s)$ in (8) is to be found. To find it we apply the transform (3)

to the boundary conditions (2a,b,c,d) and then use (8) and (9) in the resulting equations. In each case this leads to an integral equation for $A_s(s)$. To solve it we write $A_s(s)$ in the form

$$A_s(s) = A_s^{(0)}(s) + \varepsilon A_s^{(1)}(s) + \varepsilon^2 A_s^{(2)}(s) + 0(\varepsilon^3) \; . \qquad (10)$$

Then we use (10) in the integral equation, equate coefficients of corresponding powers of ε, and obtain equations for the determination of $A_s^{(0)}$, $A_s^{(1)}$ and $A_s^{(2)}$. Next we solve these equations, as is done in Appendix A for the soft boundary (case a).

We can write the result (10) for $A_s(s)$ in all four cases in the form of a suitable linear operator L acting on A_i. Explicitly we find

$$\begin{aligned} A_s(s) = [LA_i](s) = \; & \underline{+} \{ A_i(s) \\ & + 2\varepsilon \int H(s-s')m(s,s')A_i(s')ds' \\ & + 2\varepsilon^2 \int H(s-s')m(s,s') \\ & \cdot \int H(s'-s'')m(s',s'')A_i(s'')ds''ds' + 0(\varepsilon^3)\} \; . \end{aligned} \qquad (11)$$

In (11), H is the transform of h. The function m and the choice of sign are listed in Table I for the four cases. This result (11) gives the Fourier transform of the amplitude of the reflected wave for the deterministic problem.

3. Reflection coefficients in the random case.

We shall now use the preceding results to treat the case in which the function $h(x,y,t)$, describing the admittance, impedance, or the surface, is a sample function of a random process. This means that h depends upon a parameter which is a random variable having an associated probability distribution. Consequently, functionals of h, such as H, ϕ, Φ, and A_s, are random and depend upon the same random variable. We permit the incident field ψ, and hence Ψ and A_i, to be either deterministic or random. However, for the random case, we restrict our attention to ψ and h, and therefore A_i and H, which are statistically independent. Without loss of generality, we proceed as if ψ and A_i were random, since the deterministic case is the zero fluctuation limit of the random case.

The expected value or statistical mean value of a random field is called the coherent field. We use angular brackets to denote the mean or ensemble average. For example, $\langle\phi\rangle$ is the coherent scattered field. Its transform $\langle\Phi\rangle$, obtained from (8) and (11) by averaging, is

$$\begin{aligned}\langle\Phi(s,z)\rangle &= \langle A_s(s)\rangle e^{-ik(s)z} \\ &= [\langle L\rangle\langle A_i\rangle](s)e^{-ik(s)z} .\end{aligned} \tag{12}$$

The average operator $\langle L\rangle$, obtained by averaging (11), transforms a function F according to the rule

$$[\langle L\rangle F](s) = \underline{+}\{F(s)$$

$$+ 2\varepsilon \int \langle H(s-s')\rangle m(s,s')F(s')ds'$$

$$+ 2\varepsilon^2 \iint \langle H(s-s')H(s'-s'')\rangle m(s,s')m(s',s'')F(s'')ds'ds'' \tag{13}$$

$$+ 0(\varepsilon^3)\} \ .$$

We note that only second order moments of H and h are required to compute the coherent field when we neglect $0(\varepsilon^3)$.

The average of the scattered amplitude simplifies significantly if the process h is statistically second order stationary in space and time. This means that $\langle h\rangle$ is a constant, h_0, and the two-point correlation r of h depends only on the difference in arguments. Therefore, we have

$$\langle h(p)\rangle = h_0 \ , \tag{14}$$

$$\langle h(p + p')h(p')\rangle = r(p) + h_0^2 \ . \tag{15}$$

These equations imply that the first two moments of the transform H are:

$$\langle H(s)\rangle = h_0\delta(s) \ , \tag{16}$$

$$\langle H(s)H(s')\rangle = R(s)\delta(s + s') + h_0^2\delta(s)\delta(s') \ . \tag{17}$$

Here $R(s)$, the spectral power density of h, is the Fourier transform of the auto-correlation function r, and δ is the Dirac delta function. The function $R(s)$ is a positive, even, real-valued function of s.

Substituting (16) and (17) into (13) yields:

$$\langle L \rangle = C(s) I + 0(\varepsilon^3) \ . \tag{18}$$

Here I is the identity and $C(s)$ is given by

$$C(s) = \pm\Big\{1 + 2\varepsilon h_0 m(s,s) + 2\varepsilon^2 h_0^2 m^2(s,s) + 2\varepsilon^2 \int R(s-s')m(s,s')m(s',s)ds' + 0(\varepsilon^3)\Big\} \ . \tag{19}$$

From (12) and (18), we see that $C(s)$ is the ratio of the coherent scattered amplitude to the coherent incident amplitude

$$\langle A_s(s) \rangle = C(s) \langle A_i(s) \rangle + 0(\varepsilon^3) \ . \tag{20}$$

Thus $C(s)$ is the reflection coefficient of the surface. It is given in Table II for the four cases, based upon (19) and Table I. If the incident wave is a plane-wave, (20) holds for the actual wave amplitudes, as well as for the transformed amplitudes. Hence, $C(s)$ is also called the plane wave reflection coefficient.

4. Second moments of the acoustic potential.

We now use the previous results to evaluate γ, the second moment of the total acoustic potential. This two-point, two-time autocorrelation of $\psi + \phi$ is defined by

$$\gamma(x,X) = \langle[\psi + \phi](X + x/2)[\psi + \phi](X - x/2)\rangle \ . \tag{21}$$

Here $X = (X,Y,Z,T)$ is the midpoint of the two points at which γ is evaluated and $x = (x,y,z,t)$ is the difference between the two points. The transform of γ with respect to (x,y,t) with transform variable $s = (\alpha,\beta,\omega)$, and then with respect to (X,Y,T) with transform variable $q = (a,b,w)$ is denoted by Γ. We have from (8) and (9)

$$\begin{aligned}\Gamma(s,q;z,Z) &= \langle[\Psi + \Phi](q/2 + s, Z + z/2)[\Psi + \Phi](q/2 - s,\ Z - z/2)\rangle \\ &= B_{ii}(s,q)\exp i\{+ [k_+ + k_-]Z + [k_+ - k_-]z/2\} \\ &\quad + B_{is}(s,q)\exp i\{+ [k_+ - k_-]Z + [k_+ + k_-]z/2\} \\ &\quad + B_{si}(s,q)\exp i\{- [k_+ - k_-]Z - [k_+ + k_-]z/2\} \\ &\quad + B_{ss}(s,q)\exp i\{- [k_+ + k_-]Z - [k_+ - k_-]z/2\} \ .\end{aligned} \tag{22}$$

Here $k_{\pm} = k(q/2 \pm s)$.

In (22) the amplitude correlations B_{ii}, etc. are defined in terms of the scattered and incident amplitudes as follows:

$$\begin{aligned}B_{ii}(s,q) &= \langle A_i(q/2 + s)A_i(q/2 - s)\rangle \ , \\ B_{si}(s,q) &= \langle A_s(q/2 + s)A_i(q/2 - s)\rangle = \langle[\langle L\rangle A_i](q/2 + s)A_i(q/2 - s)\rangle \\ &= B_{is}(-s,q) \ , \\ B_{ss}(s,q) &= \langle A_s(q/2 + s)A_s(q/2 - s)\rangle = \langle[\langle L\rangle A_i](q/2 + s)[\langle L\rangle A_i](q/2 - s)\rangle \\ &\quad + \langle[L'A_i](q/2 + s)[L'A_i](q/2 - s)\rangle \ .\end{aligned} \tag{23}$$

The average $\langle L \rangle$ is given by (13) and the fluctuating part of L, denoted $L' \equiv L - \langle L \rangle$, is

$$[L'F](s) = \pm 2\varepsilon \int [H(s-s') - \langle H(s-s')\rangle]m(s,s')F(s')ds' + O(\varepsilon^2) \ . \tag{24}$$

Only the order ε term of L' enters the computation of B_{ss}, provided we neglect $O(\varepsilon^3)$ terms.

The expression for the second moment simplifies when h is second order stationary. Then (23) yields

$$B_{si}(s,q) = C(q/2 + s)B_{ii}(s,q) + O(\varepsilon^3) \ , \tag{25}$$

$$B_{ss}(s,q) = C(q/2 + s)C(q/2 - s)B_{ii}(s,q) + 4\varepsilon^2 \int R(s-s')m(q/2+s,q/2+s')m(q/2-s,q/2-s')B_{ii}(s',q)ds' + O(\varepsilon^3) \ . \tag{26}$$

Further simplification results if the incident field $\psi(x,y,z,t)$ is statistically second order stationary in the variables x, y, and t. For this case, we have:

$$B_{ii}(s,q) = I_i(s)\delta(q) \ , \tag{27}$$

where $I_i(s)$ is the intensity of the incident field. With (27), equations (25) and (26) become

$$B_{si}(s,q) = C(s)I_i(s)\delta(q) + O(\varepsilon^3) \ , \tag{28}$$

$$B_{ss}(s,q) = I_s(s)\delta(q) + O(\varepsilon^3) \ . \tag{29}$$

Here $I_s(s)$, the intensity of the outgoing field, is defined by

$$I_s(s) = |C(s)|^2 I_i(s) + k^{-2}(s) \int \sigma(s,s')I_i(s')ds' + O(\varepsilon^3) \ . \quad (30)$$

In (30), we have introduced the important quantity $\sigma(s,s')$, the differential scattering coefficient:

$$\sigma(s,s') = 4\varepsilon^2 k^2(s)R(s-s')|m(s,s')|^2 \ . \quad (31)$$

The scattering coefficients for admittance, impedance, soft, and hard boundary conditions appear in Table III. Since $R \geq 0$, we note that $\sigma \geq 0$.

The physical interpretation of $I_s(s)$ is evident from its definition (30). It consists of the incident intensity function $I_i(s)$ multiplied by the absolute square of the reflection coefficient for wavevector s, plus the integral over the contributions from the incident waves with wavevector s' which are scattered from s' to s. Thus it represents the sum of the reflected and scattered intensities with wavevector s.

By using the results for stationary processes in (22), and again making use of (7), we obtain the following form for Γ:

$$\begin{aligned}\Gamma(s,q;z,Z) &= \delta(q)[I_i(s)e^{ik(s)z} + \bar{C}(s)I_i(s)e^{2ik(s)Z} \\ &\quad + C(s)I_i(s)e^{-2ik(s)Z} + I_s(s)e^{-ik(s)z}] + O(\varepsilon^3) \ , \\ &\qquad (\omega/c)^2 \geq \alpha^2 + \beta^2 \ , \\ &= \delta(q)[I_i(s)e^{2ik(s)Z} + \bar{C}(s)I_i(s)e^{ik(s)z} \\ &\quad + C(s)I_i(s)e^{-ik(s)z} + I_s(s)e^{-2ik(s)Z}] + O(\varepsilon^3) \ , \\ &\qquad (\omega/c)^2 \leq \alpha^2 + \beta^2 \ .\end{aligned} \quad (32)$$

The result (32) gives the Fourier transform of the two-point two-time correlation function of the total potential $\psi + \phi$. It applies when the function h and the incident field ψ are both statistically homogeneous with respect to x, y and t. In particular it holds for a time-harmonic plane wave with a random phase which is uniformly distributed in the interval $(0,2\pi)$. It also holds for a non-random time harmonic plane wave provided that the averages denoted by angular brackets are interpreted to mean stochastic averaging and time averaging with respect to T. From (32) we see that the total potential is second order stationary with respect to the midpoint coordinates X, Y and T, but not with respect to Z.

The corresponding result for the scattered field alone is given by the term in (32) proportional to I_s. Denoting it Γ_{ss}, we have

$$\begin{aligned}\Gamma_{ss}(s,q;z,Z) &= I_s(s)\delta(q)e^{-ik(s)z} + O(\varepsilon^3)\ , \quad (\omega/c)^2 \geq \alpha^2 + \beta^2\ , \\ &= I_s(s)\delta(q)e^{-2ik(s)Z} + O(\varepsilon^3)\ , \quad (\omega/c)^2 \leq \alpha^2 + \beta^2\ .\end{aligned} \tag{33}$$

This is the result for the Fourier transform of the two-point two-time correlation of the scattered field in the stationary case. We see from the expression (30) for $I_s(s)$ that in addition to the incident intensity $I_i(s)$, Γ_{ss} involves only the magnitude of the reflection coefficient, $|C(s)|$, and the differential scattering cross-section $\sigma(s,s')$ of the surface.

5. Energy flux and energy density.

The energy flux vector $j(X)$ and the energy density $e(X)$ are defined by

$$j(X) = -\rho\partial_T(\psi + \phi)\cdot\nabla_X(\psi + \phi) \ , \tag{34}$$

$$e(X) = \frac{\rho}{2}\{c^{-2}[\partial_T(\psi + \phi)]^2 + [\nabla_X(\psi + \phi)]^2\} \ . \tag{35}$$

From the wave equation (1) satisfied by $\psi + \phi$ in a source-free region, it follows that j and e are related by the energy conservation equation

$$\partial_T e + \nabla_X\cdot j = 0 \ . \tag{36}$$

When $\psi + \phi$ is random, the average flux $\langle j\rangle$ and the average energy density $\langle e\rangle$ are defined by averaging both sides of (34) and (35). These average quantities then satisfy the averaged energy conservation equation (36). They can be expressed in terms of the two-point two-time correlation function $\gamma(x,X)$. This is shown for $\langle j\rangle$ by the following chain of equalities:

$$\begin{aligned}
\langle j(X)\rangle &= -\rho\{\langle\partial_T[\psi + \phi](X)\cdot\nabla_x[\psi + \phi](x)\rangle\}_{x=X} \\
&= -\rho\{\partial_T\nabla_x\gamma(X - x,\tfrac{1}{2}[X + x])\}_{x=X} \\
&= \rho[(\partial_t + \tfrac{1}{2}\,\partial_T)(\nabla_x - \tfrac{1}{2}\,\nabla_X)\gamma(x,X)]_{x=0} \\
&= \rho[(\partial_t\nabla_x - \tfrac{1}{4}\,\partial_T\nabla_X)\gamma(x,X)]_{x=0} \ .
\end{aligned} \tag{37}$$

The last equality of (37) is a consequence of the evenness of γ in the separation vector x. A similar analysis yields the energy density at the point X:

$$\langle e(X) \rangle = -\frac{1}{2}\rho\{[c^{-2}(\partial_t^2 - \frac{1}{4}\partial_T^2) + (\Delta_x - \frac{1}{4}\Delta_X)]\gamma(x,X)\}_{x=0} \ . \tag{38}$$

When h and ψ are statistically stationary in x, y and t, then γ is independent of X, Y and T, as we have just shown in the previous section. Therefore from (37) and (38) it follows that $\langle j \rangle$ and $\langle e \rangle$ are functions only of Z. Then the averaged form of (36) reduces to $\partial_Z \langle j_z \rangle = 0$, which shows that $\langle j_z \rangle$, the z component of $\langle j \rangle$, is constant. By using (37) for $\langle j \rangle$ and evaluating the right side at X = 0, we can write this result as

$$\begin{aligned} \langle j_z(X) \rangle &= \rho\partial_t\partial_z\gamma(0,0) \\ &= \rho \iint i\omega\partial_z\Gamma(s,q,0,0)dsdq \ . \end{aligned} \tag{39}$$

The second equality follows from the definition of the transform Γ of γ at X = 0.

We now use (32) for Γ in (39) and then use (30) to express I_s. Then we can write $\langle j_z \rangle$ in the form

$$\langle j_z(X) \rangle = -\int a(s)\rho\omega k(s) I_i(s)ds \ . \tag{40}$$

Here we have introduced the absorption coefficient a(s) defined by

$$\begin{aligned} a(s) &= 1 - |C(s)|^2 - \int_{>} [\frac{\omega'}{\omega k(s)k(s')}]\sigma(s',s)ds' + O(\varepsilon^3) \ , \\ &\qquad\qquad (\omega/c)^2 \geq \alpha^2 + \beta^2 \ , \\ &= C^*(s) - C(s) \int_{>} \frac{\omega'}{[\text{illegible}]s)k(s')}]\sigma(s',s)ds' + O(\varepsilon^3) \ , \\ &\qquad\qquad (\omega/c)^2 \leq \alpha^2 + \beta^2 \ . \end{aligned} \tag{41}$$

The integrals are over the propagating modes, $(\omega'/c)^2 > \alpha'^2 + \beta'^2$, as is indicated by the symbol >. For propagating modes, (41) shows that a(s) is that fraction of the incident energy flux which is neither reflected nor scattered.

6. Renormalization.

Some of the main results we have derived contain singularities or divergences which are physically unreasonable and mathematically inappropriate. Thus for example, the reflection coefficient C(s), the scattered intensity I_s, and the absorption coefficient a(s) all become infinite for the hard boundary at $k(s) = 0$, which corresponds to grazing reflection or scattering. This is shown by case b in Tables II and III. Furthermore $\sigma(s,s')$ becomes infinite for the soft boundary at $\omega = 0$, as is shown in Table III, case a.

Multiple scattering beyond double scattering, which has been ignored in the present analysis, is the mechanism which makes the physical coefficients finite. Mathematically multiple scattering corresponds to terms of higher power in ε, all powers of which must be included to obtain finite results. Such terms are taken into account in certain other methods

of analysis, such as the smoothing method and the equivalent method of partial summation of the power series in ε.

We shall describe another method for eliminating the singularities and obtaining finite results which are physically meaningful. It is based upon representing the reflection coefficient C(s) in the following well-known admittance-impedance form:

$$C(s) = \pm \frac{1 + Q(s)}{1 - Q(s)} \quad . \tag{42}$$

We choose the sign as in Table I. To determine the function Q(s) we require that the power series expansion of C(s) given by (42) agree with our previous result (19) up to $O(\varepsilon^3)$. This yields

$$Q(s) = \varepsilon h_0 m(s,s) + \varepsilon^2 \int R(s-s')m(s,s')m(s',s)ds' + O(\varepsilon^3) \quad . \tag{43}$$

The value of Q(s) for each of the four cases is shown in Table IV.

We shall call (42) the renormalized form of C(s). It does not have the defects of the unrenormalized form (19). The appearance of the denominator $|1-Q(s)|^2$ in (41) when (42) is used in it, suggests that we should renormalize $\sigma(s,s')$ by writing

$$\sigma(s,s') = \frac{4\varepsilon^2 R(s-s')|m(s,s')|^2 k^2(s)}{|1-Q(s')|^2} \quad . \tag{44}$$

This form of σ is also free of the defects of the unrenormalized form (31). When the renormalized C and σ are used in the expression (41), it yields

the renormalized absorption coefficient a.

The renormalized expression (42) for C suggests that the coherent field satisfies an admittance or an impedance boundary condition on the mean surface. To show this, we define the effective specific admittance of the surface to be $Q^{\pm 1}(s)ck(s)/\omega$, and we denote its inverse Fourier transform by $\eta(x,y,t)$. The sign is as in Table I. Then we consider the boundary condition

$$c\partial_z\langle\psi+\phi\rangle + (2\pi)^{-3}\eta(x,y,t) * \partial_t\langle\psi+\phi\rangle = 0 \qquad \text{on } z = 0. \qquad (45)$$

Here * denotes convolution in x, y and t. This boundary condition yields exactly the plane wave reflection coefficient C given by (42).

Alternatively we can introduce the reciprocal of the admittance, which is just the effective specific impedance of the surface, and is given by $Q^{\mp 1}(s)\omega/ck(s)$. We denote its inverse transform $\mu(x,y,t)$. Then instead of (45) we write

$$\partial_t\langle\psi+\phi\rangle + c(2\pi)^{-3}\mu(x,y,t) * \partial_z\langle\psi+\phi\rangle = 0 \qquad \text{on } z = 0. \qquad (46)$$

Again (46) leads to (42). The value of (45) and (46) is that they can be used for a rough surface which is not nearly flat, provided that its radius of curvature is large compared to the correlation length of the roughness and to the wavelength.

There are plane wave solutions of (1) which satisfy (45) or (46) with no incident wave, and which decay with increasing distance z away from the surface. Substitution of a plane wave into (45) or (46)

yields the condition $Q(s) = 1$. This shows that such waves correspond to poles of the renormalized reflection coefficient $C(s)$ given by (42). The roots of $Q(s) = 1$ have been examined for the hard surface, with $Q(s)$ given by case b in Table IV, by Wenzel (3), and for this and other cases by Watson, Stickler and Keller (4).

The renormalization method is arbitrary in the sense that the form (42) of $C(s)$ is postulated. Once this is done however, the value of $Q(s)$ is determined through terms of order ε^2. Similarly, the form (44) of σ is specified. As a consequence the renormalized absorption coefficient for motionless hard and soft surfaces vanishes through terms of order ε^2, instead of vanishing identically.

7. Embossed surfaces and Twersky's theory

Some rough surfaces can be viewed as flat surfaces with random collections of bumps or bosses glued onto them. This is the model of a rough surface employed by Twersky [1] in his extensive investigations. Because it is so different from that employed in the application of regular perturbation theory, the relation between perturbation theory and Twersky's theory has never been adequately explained. We shall explain it by applying our results to embossed planes and then comparing them with Twersky's results. In doing so we shall extend the model to moving surfaces by permitting the bosses to move.

We begin by writing the equation of the surface as $z = \varepsilon h(x,y,t)$ with

$$h(x,y,t) = \sum_{i=1}^{N} b[(x-x_i)\cos\theta - (y-y_i)\sin\theta_i,\ (y-y_i)\cos\theta + (x-x_i)\sin\theta_i]\,. \tag{47}$$

Here $b(x,y)$ determines the shape of a single boss located at the origin, and N is the total number of bosses.

All of the bosses are of the same shape, the i^{th} one having been rotated through the random angle θ_i and translated to the random position $r_i = (x_i, y_i)$ at time t. Thus the statistics of the surface are specified by the joint probability distribution of all the $r_i = (x_i, y_i)$ and θ_i, which are random functions of t. However for our purposes it will suffice to specify the probability distribution for one or two bosses at two times, which we shall now do.

Let us denote by $p_2(r_i, \theta_i, t; r_j', \theta_j', t')$ the probability density for boss i to be at r_i with orientation θ_i at time t, and for boss $j \neq i$ to be at r_j' with orientation θ_j' at time t'. Similarly let $p_1(r_i, \theta_i, t; r_i', \theta_i', t')$ be the probability density for boss i to be at r_i with orientation θ_i at time t, and to be at r_i' with orientation θ_i' at time t'. For distributions statistically stationary in space and time, these densities must have the forms

$$p_2 = A^{-2} w_2(r_i - r_j', t-t', \theta_i, \theta_j') \quad r_i \text{ and } r_j \text{ in } D, \tag{48}$$

$$p_1 = A^{-1} w_1(r_i - r_i', t-t', \theta_i, \theta_i') \quad r_i \text{ and } r_i' \text{ in } D\,. \tag{49}$$

Here D is the domain throughout which the bosses are distributed and A is its area. Ultimately we will let D become the whole plane with A and N tending to infinity with a fixed ratio $N/A = \nu$. Thus ν is the average number of bosses per unit area.

To calculate $\langle h \rangle = h_0$ we need the probability density for particle i to be at r_i with orientation θ_i at time t, which we can obtain by setting $t=t'$ in p_1. From the definition of p_1 it follows that at $t=t'$, w_1 in (49) must have the form $w_1(r_i - r_i', 0, \theta_i, \theta_i') = \delta(r_i - r_i')\delta(\theta_i - \theta_i')f(\theta_i)$ for r_i in D. Then from (47) we get

$$h_0 = \langle h \rangle = NA^{-1}\iiint_D b_i dx_i dy_i f(\theta_i) d\theta_i = \nu v \int f(\theta_i) d\theta_i = \nu v \; . \qquad (50)$$

Here $b_i(r)$ is the summand in (47), and v, the integral of b with respect to x and y, is the volume of a boss. Next we use (47)-(49) to calculate $r(p)$, the auto-correlation function of h, as follows:

$$r(p) = \langle h(r+r',t+t')h(r',t')\rangle - h_0^2 = \int_0^{2\pi}\int_0^{2\pi}\iint_{DD}[\nu^2 w_2(r_i-r_j,t,\theta_i,\theta_j)$$
$$+ \nu w_1(r_i-r_j,t,\theta_i,\theta_j)]b_i(r+r')b_j(r')dr_i dr_j d\theta_i d\theta_j - \nu^2 v^2 \; . \qquad (51)$$

From (51) we can calculate the spectral power density $R(s)$, which is the Fourier transform of $r(p)$, by noting that the integrals in (51) are convolutions with respect to r_i and r_j. Then we obtain

$$R(s) = (2\pi)^4 \int_0^{2\pi}\int_0^{2\pi} [\nu^2 W_2(s,\theta_i,\theta_j)+\nu W_1(s,\theta_i,\theta_j)]B_i(\alpha,\beta)\bar{B}_j(\alpha,\beta)d\theta_i d\theta_j - \nu^2 v^2 \delta(s), \qquad (52)$$

where $B_i(\alpha,\beta) = B[\alpha\cos\theta_i + \beta\sin\theta_i, - \alpha\sin\theta_i + \beta\cos\theta_i]$. Here B is the two dimensional transform of $b(x,y)$. The results (50) for h_0 and (52) for $R(s)$ can be used in (42)-(44) to get C and σ.

The integrals in (52) simplify when the bosses are similarly aligned, when the bosses have uniformly distributed orientations, and when they are axially symmetric. For example in the first case, when all the bosses are permanently aligned in the direction $\theta = 0$, we take $W_k(s,\theta_i,\theta_j) = \tilde{W}_k(s)\delta(\theta_i)\delta(\theta_j)$, $k = 1,2$. Then (52) becomes

$$R(s) = (2\pi)^4[\nu^2\tilde{W}_2(s)+\nu\tilde{W}_1(s)]\,|B(\alpha,\beta)|^2 - \nu^2 v^2 \delta(s) \; . \qquad (53)$$

We shall now compare our results for an embossed surface with an old

result of Twersky [1]. His results apply to permanently aligned bosses, so we must use (53) for $R(s)$. He also considers small number density ν, so we must omit the ν^2 term from (53). Furthermore his surfaces are not moving, so we must set $(2\pi)^2\tilde{W}_1(s) = \delta(\omega)$. With these simplifications, (53) becomes

$$R(s) = (2\pi)^2\nu|B(\alpha,\beta)|^2\delta(\omega) + 0(\nu^2) \ . \tag{54}$$

When we use (54) and (50) in (43) and (44) we obtain

$$\begin{aligned} Q(s) = \varepsilon\nu\nu m(s,s)+\varepsilon^2(2\pi)^2\nu \int |B(\alpha-\alpha',\beta-\beta')|^2 m(s,s')m(s',s)\delta(\omega-\omega')ds' \\ + 0(\varepsilon^3) + 0(\nu^2) \end{aligned} \tag{55}$$

$$\sigma(s,s') = \frac{4\varepsilon^2(2\pi)^2\nu|B(\alpha-\alpha',\beta-\beta')|^2|m(s,s')|^2k^2(s)}{|1-Q(s')|^2}\delta(\omega-\omega')+0(\varepsilon^3)+0(\nu^2) \tag{56}$$

In Appendix B we calculate to second order in ε the scattering amplitude $f(u,e)$ of a single boss on a plane, for the incident wave vector e and the scattered wave vector u. We shall denote the result f_B because it is the second Born approximation. It is given by (B6). In terms of f_B, we can write (55) and (56) in the forms

$$Q(s) = \frac{\nu\pi f_B(\hat{e},e)}{(\omega/c)k(s)} \tag{57}$$

$$\sigma(s,s') = \frac{\nu}{(\omega/c)^2} \cdot \left|\frac{f(u,e)}{1-Q(s')}\right|^2 \delta(\omega-\omega') \ . \tag{58}$$

Here $\hat{e} = (e_1,e_2,-e_3)$, while the relation between s and s', and u and e is given by (B7) and (B8).

These results (58) and (59) are of exactly the same form as those in the abstract of Twersky [1] with f_B instead of the exact scattering amplitude f. Therefore the present results for the renormalized reflection and differential scattering coefficients reduce to those of Twersky when they are specialized to hard and soft embossed planes in which the bosses are permanently aligned, not moving, and of low number density. Then they yield Twersky's results with f replaced by f_B. When f_B is not a good approximation to f, as is the case for bosses high compared to a wavelength or bosses having steep slopes, Twersky's results are more accurate than the present ones. However the present results include surfaces which are not embossed planes, which are moving, which may have impedance or admittance boundary conditions, and which can have a high density of bosses. In addition they yield the two-point two-time correlation function of the field, not just the coherent field and the intensity.

Twersky [5] has extended his theory to non-aligned bosses and to surfaces with a high density of bosses. We plan to compare the present results with these newer results of his.

Appendix A. Solution for $A_s(s)$.

To find $A_s(s)$ for the case of a soft boundary, we take the Fourier transform of (2a) to get

$$i\omega[\Phi(s,0) + \Psi(s,0)] + i\varepsilon \int H(s-s')\omega'[\partial_z\Phi(s',0) + \partial_z\Psi(s',0)]ds'$$
$$+ \frac{i\varepsilon^2}{2} \int H(s-s') \int H(s'-s'')\omega''[\partial_z^2\Phi(s'',0) + \partial_z^2\Psi(s'',0)]ds''ds'$$
$$+ 0(\varepsilon^3) = 0 . \tag{A1}$$

Then we use (8) and (9) for A_s and A_i in (A1) and obtain

$$i\omega[A_s(s) + A_i(s)] + \varepsilon \int H(s-s')\omega'k(s')[A_s(s') - A_i(s')]ds'$$
$$- \frac{i\varepsilon^2}{2} \int H(s-s') \int H(s'-s'')\omega''k^2(s'')[A_s(s'') + A_i(s'')]ds''ds'$$
$$+ 0(\varepsilon^3) = 0 . \tag{A2}$$

Now we expand A_s in powers of ε in the form

$$A_s(s) = A_s^{(0)}(s) + \varepsilon A_s^{(1)}(s) + \varepsilon^2 A_s^{(2)}(s) + 0(\varepsilon^3) . \tag{A3}$$

Upon using (A3) in (A2), and equating to zero the coefficient of each power of ε, we get the equations

$$i\omega[A_s^{(0)}(s) + A_i(s)] = 0 , \tag{A4}$$

$$i\omega A_s^{(1)}(s) + \int H(s-s')\omega'k(s')[A_s^{(0)}(s') - A_i(s')]ds' = 0 , \tag{A5}$$

$$i\omega A_s^{(2)}(s) + \int H(s-s')\omega'k(s')A_s^{(1)}(s')ds'$$
$$- \frac{i}{2} \int H(s-s') \int H(s'-s'')\omega''k^2(s'')[A_s^{(0)}(s'') + A_i(s'')]ds'ds'' = 0 . \tag{A6}$$

From (A4) we get

$$A_s^{(0)}(s) = -A_i(s) \ . \tag{A7}$$

Then (A5) yields

$$A_s^{(1)}(s) = 2i\omega^{-1} \int H(s-s')\omega' k(s')A_i(s')ds' \ . \tag{A8}$$

Next (A6) has the solution

$$A_s^{(2)}(s) = -2\omega^{-1} \int H(s-s')k(s') \int H(s'-s'')\omega'' k(s'')A_i(s'')ds''ds' \ . \tag{A9}$$

By substituting (A7)-(A9) into (A3) we obtain the expansion of $A_s(s)$. This solution is of the form (11) with the minus sign, and with $m(s,s') = (i\omega'/\omega)k(s')$.

Appendix B. Scattering Amplitude

We consider a boss located on the xy-plane and insonified by an incident plane wave in the direction of the unit vector $e = [e_1,e_2,e_3]$, where $e_3 < 0$. The incident field is given by

$$\psi(x,y,z,t) = \exp[(2\pi i/\lambda)(e_1x+e_2y+e_3z-ct)], \tag{B1}$$

where λ is the wavelength. The scattered field ϕ has the following expansion for field points far from the boss

$$\phi(x,y,z,t) = \pm\exp[(2\pi i/\lambda)(e_1x+e_2y-e_3z-ct)] \pm f(u,e)\ h_0(2\pi[x^2+y^2+z^2]^{\frac{1}{2}}/\lambda)\cdot \exp(-2\pi ict/\lambda) + O(\lambda/[x^2+y^2+z^2]^{\frac{1}{2}}) \tag{B2}$$

The plus sign applies to a hard boundary and the minus sign to a soft one.

The first term in (B2) is the specular reflection of an incident plane wave from the xy-plane. The second term, which is the wave scattered from the boss, is proportional to

$$h_0(r) = \exp(ir)/(ir) , \tag{B3}$$

and to the scattering amplitude $f(u,e)$. Here $u = [x,y,z]/[x^2+y^2+z^2]^{\frac{1}{2}}$ is the unit vector in the direction of observation.

If the height of the boss is small compared to a wavelength, and its slope is small, then a Born expansion can be used to compute its scattering amplitude. We denote the second-order Born approximation of f by f_B. To compute it we use the formulas given in Sec. 2 with the incident field given by (B1) and the surface shape $h(x,y,t) = b(x,y)$. From the transforms of these quantitites and equations (7) and (9), we have

$$A_i(s) = \delta(\alpha-2\pi e_1/\lambda)\cdot\delta(\beta-2\pi e_2/\lambda)\delta(\omega+2\pi c/\lambda) \tag{B4}$$

and

$$H(s) = B(\alpha,\beta)\delta(\omega) . \tag{B5}$$

We substitute (B4) and (B5) into (11) to get an expression for the amplitude of the scattered field transform $A_s(s)$. Then we obtain the transform of the scattered field by using this result in (8).

We invert this transform for large values of $([x^2+y^2+z^2]^{\frac{1}{2}}/\lambda)$ by using the method of stationary phase. This yields an expansion of ϕ of the form (B2). Upon comparing terms, we get

$$f_B(u,e) = 4\pi(\omega/c)k(s)\cdot\{\varepsilon B(\alpha-\alpha',\beta-\beta')m(s,s')+\varepsilon^2\iint B(\alpha-\alpha'',\beta-\beta'')B(\alpha''-\alpha',\beta''-\beta') m(s,s'')m(s'',s')d\alpha''d\beta''\} . \tag{B6}$$

The right side of (B6) is evaluated with

$$s = [\alpha,\beta,\omega] = (2\pi/\lambda)[u_1,u_2,-c] \tag{B7}$$

and

$$s' = [a',\beta',\omega'] = (2\pi/\lambda)[e_1,e_2,-c] \ . \tag{B8}$$

In (B6), $k(s)$ is defined by (7) and m is given in Table I for both the hard and soft boundary conditions.

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Parameter	(a) Soft	(b) Hard	(c) Admittance	(d) Impedance
$\pm$	$-$	$+$	$+$	$-$
$m(s,s')$	$i(\frac{\omega'}{\omega})k(s')$	$ik^{-1}(s)[(\frac{\omega'}{c})^2 - \alpha\alpha' - \beta\beta']$	$(\omega'/c)k^{-1}(s)$	$(\omega/c)^{-1}k(s')$

TABLE I. The sign and the function $m(s,s')$.

(a) Soft: $-1 - 2i\varepsilon h_0 k(s) + 2[\varepsilon h_0 k(s)]^2 + 2\varepsilon^2 k(s) \int R(s-s')k(s')ds'$

(b) Hard: $1 + 2i\varepsilon h_0 k(s) - 2[\varepsilon h_0 k(s)]^2 - 2\varepsilon^2 k^{-1}(s) \int R(s-s')k^{-1}(s')$

$[(\omega/c)^2 - \alpha\alpha' - \beta\beta'] \cdot [(\omega'/c)^2 - \alpha\alpha' - \beta\beta']ds'$

(c) Admittance: $1 + 2\varepsilon h_0 \omega/ck(s) + 2[\varepsilon h_0 \omega/ck(s)]^2 + \frac{2\varepsilon^2\omega}{c^2 k(s)} \int \frac{\omega' R(s-s')}{k(s')} ds'$

(d) Impedance: $-1 - 2\varepsilon h_0 ck(s)/\omega - 2[\varepsilon h_0 ck(s)/\omega]^2 - \frac{2\varepsilon^2 c^2 k(s)}{\omega} \int \frac{k(s')R(s-s')}{\omega'} ds'$

TABLE II. Reflection coefficient $C(s)$ through order ε^2.

(a) Soft: $4\varepsilon^2 R(s-s')(\frac{\omega'}{\omega})^2|k(s')|^2 k^2(s)$

(b) Hard: $4\varepsilon^2 R(s-s')[\omega'/c)^2 - \alpha\alpha' - \beta\beta']^2 \text{sgn } k^2(s)$

(c) Admittance: $4\varepsilon^2 R(s-s')(\omega'/c)^2 \text{sgn } k^2(s)$

(d) Impedance: $4\varepsilon^2 R(s-s')(c/\omega)^2|k(s')|^2 k^2(s)$

TABLE III. Differential scattering coefficient $\sigma(s,s')$.

(a) Soft: $i\varepsilon h_0 k(s) - \varepsilon^2 k(s) \int R(s-s')k(s')ds'$

(b) Hard: $i\varepsilon h_0 k(s) - \varepsilon^2 k^{-1}(s) \int R(s-s')k^{-1}(s')[(\omega/c)^2 - \alpha\alpha' - \beta\beta'][(\omega'/c)^2 - \alpha\alpha' - \beta\beta']ds'$

(c) Admittance: $\varepsilon h_0 \omega/ck(s) + \frac{\varepsilon^2 \omega}{c^2 k(s)} \int \frac{\omega' R(s-s')}{k(s')} ds'$

(d) Impedance: $\varepsilon h_0 ck(s)/\omega + \frac{\varepsilon^2 c^2 k(s)}{\omega} \int \frac{k(s')R(s-s')}{\omega'} ds'$

TABLE IV. $Q(s)$, which occurs in the renormalized reflection and scattering coefficients, through order ε^2.